

Week 6 Worksheet - Differential Equations

Instructions. Follow the instructions of your TA and do the following problems. You are not expected to finish all the problems. So take your time! :)

Two Types of 1st order differential equations to solve in this course:

1. Separable
2. Linear

1. Determine the type of the differential equations (*separable or linear*).

(a) $\frac{dy}{dx} + y \tan x = \cos x$ linear

(b) $\frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = \cos x$ sep.

(c) $(1+x^2) \frac{dy}{dx} + 2xy = 3(1+x^2)$ linear.

(d) $\frac{dy}{dx} - e^x y^2 = e^x$ sep.

(e) $\frac{dA}{dt} = 3 - \frac{2A}{20-t}$ linear.

(f) $\frac{dy}{dx} = 4x^3(y + e^{x^4})$ linear

(g) $\frac{dy}{dx} = 6x^5 + y^2 6x^5$ sep

(h) $\frac{dP}{dt} - (1000 - P)P = 0$ linear.

2. Some understanding of the linear differential equations

$$\frac{dy}{dx} + a(x)y = b(x).$$

(a) Denote $A(x) = \int a(x) dx$. What is $\frac{d}{dx} A(x)$?

(b) Compute $\frac{d}{dx}(e^{A(x)}y)$ using the product rule, where y is a function of x .

(a) $\frac{d}{dx} A(x) = a(x)$

(b) $\frac{d}{dx}(e^{A(x)})y + e^{A(x)}\frac{dy}{dx}$

$$= e^{A(x)}a(x)y + e^{A(x)}\frac{dy}{dx}$$

(c) Fill in the brackets:

Multiply the linear equation by $e^{A(x)}$ on both sides

$$e^{A(x)} \frac{dy}{dx} + e^{A(x)}a(x)y = e^{A(x)}b(x).$$

$$\frac{d}{dx}[e^{A(x)}y] = e^{A(x)}b(x)$$

We call $m(x) = e^{A(x)} = e^{\int a(x)dx}$

the integrating factor.

$$\Leftrightarrow \frac{d}{dx}[m(x)y] = m(x)b(x)$$

Integrate both sides, we get

$$e^{A(x)}y = \int e^{A(x)}b(x)dx \Leftrightarrow y = \frac{1}{m(x)} \int m(x)b(x)dx$$

Linear 3. Solve the following differential equation.

$$\cos^2 x \frac{dy}{dx} = -y + 10, \quad y(0) = 0$$

$\frac{dy}{dx} + \sec^2 x y = 10 \sec^2 x$

$a(x) = \sec^2 x \quad \int \sec^2 x dx = \tan x + C$

$m(x) = e^{\tan x}$

$e^{-\tan x} \frac{dy}{dx} + e^{-\tan x} \sec^2 x y = 10 \sec^2 x e^{-\tan x}$

$\frac{d}{dx} [e^{-\tan x} y] = 10 \sec^2 x e^{-\tan x}$

$e^{-\tan x} y = 10 \int \sec^2 x e^{-\tan x} dx \stackrel{u=\tan x, du=\sec^2 x dx}{=} 10 \int e^u du = 10 e^{\tan x} + C$

plug in
 $\Rightarrow (x, y) = (0, 0)$

$0 = 10 + C \Rightarrow C = -10$

$e^{\tan x} y = 10 e^{\tan x} - 10$

$\Rightarrow y = \frac{10 e^{\tan x} - 10}{e^{-\tan x}} \quad \text{or } y = 10 - 10 e^{-\tan x}$

Linear 4. Solve

$$(1+x^2) \frac{dy}{dx} + 2xy = 3(1+x^2), \quad y(1) = \frac{5}{2}$$

$\frac{dy}{dx} + \frac{2x}{1+x^2} y = 3$

$a(x) = \frac{2x}{1+x^2} \quad \int \frac{2x}{1+x^2} dx \stackrel{u=1+x^2, du=2x dx}{=} \int \frac{du}{u} = \ln|u| + C = \ln(1+x^2) + C$

$m(x) = e^{\ln(1+x^2)} = 1+x^2$

$\frac{d}{dx} [(1+x^2)y] = 3(1+x^2)$

$(1+x^2)y = \int 3+3x^2 dx = 3x+x^3+C$

Plug in $(x, y) = (1, \frac{5}{2})$

$2 \cdot \frac{5}{2} = 3 + 1 + C \Rightarrow C = 1$

$y = \frac{3x+x^3+1}{1+x^2}$

Separable 5. Solve for $B(t)$ that satisfies

$$\frac{dB}{dt} = 50B(1-B), \quad B(0) = 2.$$

(This problem might be helpful for hw page 60 #5)

$\frac{dB}{B(1-B)} = 50 dt$

$\frac{1}{B(1-B)} = \frac{A}{B} + \frac{B}{1-B}$

$1 = A(1-B) + CB$

Plug in $B=0 \Rightarrow \begin{cases} A=1 \\ B=1 \end{cases}$ and $C=1$

$\frac{1}{B(1-B)} = \frac{1}{B} + \frac{1}{1-B} = \frac{1}{B} - \frac{1}{B-1}$

$\Rightarrow \boxed{\frac{B}{B-1} \ln|B-1|}$

$\ln|B| - \ln|B-1| = 50t + C$

$\ln\left|\frac{B}{B-1}\right| = 50t + C$

$\left|\frac{B}{B-1}\right| = e^{50t} \cdot e^C$

$\frac{B}{B-1} = \underbrace{\pm e^C}_{\text{denote as } A, \text{ some constant.}} e^{50t}$

$-\frac{B}{B-1} = Ae^{50t}$

Plug in $t=0, B=2$

$2 = Ae^0 \Rightarrow A=2$

$\frac{B}{B-1} = 2e^{50t}$

$B = 2e^{50t} B - 2e^{50t}$

$(-2e^{50t})B = -2e^{50t}$

$\Rightarrow B = \frac{-2e^{50t}}{1-2e^{50t}}$